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Smooth Phase in the One-Dimensional Discrete Gaussian Model with $1/(i-j)^2$ Interaction at Inverse Temperature $\beta > 1$

Domingos H. U. Marchetti[1]

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We consider the one-dimensional discrete Gaussian model with interaction energy g satisfying $g(i, j) = g(i-j) \sim 1/(i-j)^2$ and prove that for the inverse temperature $\beta > 1$ this system displays a smooth phase characterized by $\langle (n_{x_0} - n_{y_0})^2 \rangle \leqslant C < \infty$ if the nearest neighbor coupling $g(1)$ is sufficiently large. Our method also allows us to treat the $1/(i-j)^2$ Ising model and reproves the existence of spontaneous magnetization under the above conditions.

KEY WORDS: Smooth phase; critical temperature; multiscale analysis, Peierls expansion.

1. INTRODUCTION

We consider the one-dimensional discrete Gaussian model with interaction energy $g(i, j)$ given by a positive function satisfying

$$g(i, j) = g(i-j) \sim \frac{1}{(i-j)^2} \qquad \text{as} \quad |i-j| \to \infty$$

A configuration of this model is a function $n = \{n_j\}_{j \in \mathbf{Z}}$, where $n_j \in \mathbf{Z}$ represents the height of a interface at j. To each configuration the energy H_Λ is given by

$$H_\Lambda(n) = \frac{1}{2} \sum_{i, j} g(i, j)(n_i - n_j)^2 \tag{1.1}$$

[1] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4K1, Canada.

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where we impose the boundary condition

$$n_j = 0 \qquad \text{for any} \quad j \notin \Lambda \tag{1.2}$$

with Λ being a finite interval in $\mathbf{Z}$.

The equilibrium state of this system is defined by the Gibbs measure μ_Λ on the space of all configurations

$$\mu_\Lambda(n) = \frac{1}{Z_\Lambda} e^{-\beta H_\Lambda(n)} \tag{1.3}$$

where

$$Z_\Lambda = \sum_n e^{-\beta H_\Lambda(n)} \tag{1.4}$$

is the partition function. Expectations with respect to this measure will be denoted by

$$\langle \cdot \rangle_\Lambda = \sum_n \cdot \, \mu_\Lambda(n)$$

and by $\langle \cdot \rangle = \lim_{\Lambda \to \infty} \langle \cdot \rangle_\Lambda$ we mean their thermodynamic limit. The limit exists by correlation inequalities.[7]

We will be interested in the behavior of the correlation functions, so let us introduce the external height expectation defined by

$$\langle e^{n(h)} \rangle_\Lambda = \frac{Z_\Lambda(h)}{Z_\Lambda} \tag{1.5}$$

where

$$Z_\Lambda(h) = \sum_n e^{n(h)} e^{-\beta H_\Lambda(n)} \tag{1.6}$$

is the external height partition function, $n(h) = \sum_{k \in \mathbf{Z}} n_k h_k$, and h is the external height density typically given as follows.

1. The one-point external height density

$$h_k = h^0 \delta_{k, x_0}$$

2. The two-point external height density

$$h_k = h^0(\delta_{k, x_0} - \delta_{k, y_0})$$

with x_0, $y_0 \in \Lambda$ and $h^0 \in \mathbf{R}$.

This model has been recently examined by Fröhlich and Zegarlinski[1] in connection with the localization of a quantum mechanical particle in a one-dimensional periodic potential coupled to a quantum $1/f$-noise. In that paper they established a phase transition in the sense that there exist $0<\underline{\beta}\ll\bar{\beta}<\infty$, depending on the interaction energy g, such that for the inverse temperature $\beta<\underline{\beta}$ the discrete Gaussian displays a rough phase with

$$\langle (n_{x_0}-n_{y_0})^2\rangle \geqslant C\log|x_0-y_0| \tag{1.7}$$

and for the inverse temperature $\beta>\bar{\beta}$ there exists a smooth phase characterized by

$$\langle (n_{x_0}-n_{y_0})^2\rangle \leqslant C' \tag{1.8}$$

where C, $C'>0$ are β-dependent finite constants.

Moreover, they showed (Proposition 3.1 in ref. 1) that for $\beta>\bar{\beta}$ there exist positive constants $\bar{h}=\bar{h}(\beta)$ and C such that for the one-point external height density h satisfying $0<h^0<\bar{h}$,

$$\langle e^{n(h)}\rangle \leqslant e^{h^0}(1-e^{-(\beta C-2h^0)})^{-1} \tag{1.9}$$

which implies that the moments of a discrete Gaussian measure $\mu(n)$, in the low-temperature phase, are bounded by

$$\langle (n_{x_0})^{2r}\rangle \leqslant (2r)!\,[A(\beta)]^r \tag{1.10}$$

where $A>0$ is a finite constant (odd moments are zero by $n\to -n$ symmetry).

In this paper we retrieve the low-temperature results of Fröhlich and Zegarlinski and prove that the inverse temperature below which we get a smooth phase is at most 1 in the high-$g(1)$ limit, where $g(1)$ is the nearest-neighbor interaction energy. We also obtain in this regime an upper bound on the external height expectation which leads the $2r$th moment (1.10) to be bounded by a constant to a power r times $r!$ (instead of $2r!$) and the two-point correlation function (1.8) to be finite.

More precisely, we have the following result.

Theorem 1.1. (a) Let h be the one-point external height density such that $|h^0|<\beta g(1)$. Then for any inverse temperature $\beta>1$ there exist finite positive constants $\bar{g}=\bar{g}(\beta)$ and $\theta=\theta(\beta)$ such that if $g(1)>\bar{g}$, we have

$$\langle e^{n(h)}\rangle \leqslant \exp\{\theta e^{-\beta g(1)}\cosh h^0\} \tag{1.11}$$

(b) Let h be the two-point external height density such that $|h^0| < \beta g(1)$. Under the above assumptions, there exists $\delta = \delta(\beta) > 0$ such that

$$\langle e^{n(h)} \rangle \leqslant \exp\{2\theta e^{-\beta g(1)} \cosh h^0\} \exp\{\theta e^{-\beta g(1)} \cosh h^0 \, |x_0 - y_0|^{-\delta}\} \tag{1.12}$$

Notice that the right-hand sides of (1.11) and (1.12) are analytic functions on h^0. One can differentiate both sides at $h^0 = 0$ to obtain the following corollary:

Corollary 1.2. Let $\beta > 1$ and $\bar{g}$ as above. Then if $g(1) > \bar{g}$, we have

$$\langle (n_{x_0})^{2r} \rangle \leqslant C^r r!$$

for any $r \in \mathbf{N}$, and

$$\langle (n_{x_0} - n_{y_0})^2 \rangle \leqslant C'$$

where C and C' are finite constants.

Remark. The method we use to prove Theorem 1.1 is also suitable to study spontaneous symmetry breaking of the $1/(i-j)^2$ Ising model. We consider the Ising Hamiltonian given by (1.1) with $n = \{n_j = \pm 1\}_{j \in \mathbf{Z}}$ and boundary condition $n_j = 1$ for all $j \notin \Lambda$ and show (details in Section 4) that for any inverse temperature $\beta > 1$ there exist $\theta = \theta(\beta) < \infty$ such that

$$\langle 1 - n_{x_0} \rangle \leqslant \theta e^{-\beta g(1)} < 1 \tag{1.13}$$

provided $g(1)$ is sufficiently large. Spontaneous magnetization in the Ising chain with $1/(i-j)^2$ interaction energy was proved by Fröhlich and Spencer[2] for the inverse temperature large enough. Imbrie and Newman[2] have proven (1.13), among other results, in the conditions under which we have stated it.

To prove Theorem 1.1, we modify the procedure in the Fröhlich–Zegarlinski proof. In the proof of (1.8) they extended the Peierls argument developed in ref. 2 for the $1/(i-j)^2$ Ising chain in order to control expectations of unbounded variables. We here use an alternative procedure to handle this problem. As in refs. 4 and 5, expectations in the discrete Gaussian chain are written as a convex combination of expectations in diluted gases of "neutral" jump sequences of variable sizes. We then apply a standard Peierls argument to each term of this expansion. This goal is accomplished by following closely the treatment given by Marchetti *et al.*[5] (see also ref. 6) in the study of the external charge correlation functions of the two-dimensional Coulomb gas.

Our expansion consists in initially rewriting both partition functions (1.4) and (1.6) as a convex combination of (appropriately defined) regular partition functions in a given initial scale (Theorem 2.2). It is then proven that *regular* partition functions at a given scale can be written as a convex combination of *regular* partition functions at the next scale (Lemma 3.2).

The scales we use are of the form $d_{k+1} \cong d_k^\alpha$ with $1 < \alpha < 2$. At scale N, where N is such that $d_{N-1} < |\Lambda| \leqslant d_N$, each of these *regular* partition functions is characterized by a collection $\mathcal{N} = \{J\}$ of disjoint ordered sequences of jumps J, whose sizes vary from the initial scale up to the last scale N. This collection is such that:

(i) Any $J \in \mathcal{N}$ is neutral, i.e., is a jump sequence which starts and finishes at the same height.

(ii) All J are weighted by an activity $\zeta(J)$.

(iii) $\mathcal{N}$ satisfies an appropriately defined sparse condition.

Items (i)–(iii) play an important role in describing the low-temperature phenomenon. Because of neutrality, jump sequences which contribute to the external height expectation $\langle e^{n(h)} \rangle$ are essentially those in the subset $\chi \subset \mathcal{N}$ of J whose support "overlaps" the support of h (Lemma 4.1). Taking, for example, h to be the two-point density, under the *sparse* condition, χ has at most two jump sequences in each scale and $\sum_{J \in \chi} \zeta(J)$ is finite independently of N and the distance $|x_0 - y_0|$. This means that typical configurations in the discrete Gaussian chain are smooth in the region of parameters where the expansion is valid.

This paper is organized as follows. In Section 2 the partition function of the discrete Gaussian chain is rewritten as a convex combination of *regular* partition functions. This is the first step in the inductive procedure in Section 3. In Section 4 we perform a Peierls argument and prove Theorem 1.1. We consider the main contribution of this paper to be the possibility of treating the $1/(i-j)^2$ discrete Gaussian model as well as the $1/(i-j)^2$ Ising model within the same framework.

2. FIRST STEP

Following ref. 5, we start by rewriting the partition function (1.4) as a convex combination of "regular" partition functions at the first scale.

Notice that any configuration n_Λ satisfying the boundary condition (1.2) specifies a *unique* sequence of jumps $J_\Lambda = J(n_\Lambda) = \{J_i\}_{i \in \Lambda^*}$, where for each $i \in \Lambda^*$, J_i is the difference between two consecutive heights, i.e.,

$$J_i \Leftrightarrow n_{i+1/2} - n_{i-1/2} \equiv dn_i$$

and Λ^* is the interval in the dual lattice $\mathbf{Z}^*$ given by

$$\Lambda^* = \{j+1/2\}_{j\in\Lambda} \cup \{j-1/2\}_{j\in\Lambda}$$

Let $\mathscr{J}_\Lambda$ be the set of all jump functions J_Λ as above, i.e.,

$$\mathscr{J}_\Lambda = \{J: i\in\mathbf{Z} \to J_i \in \mathbf{Z}: J_j = 0 \text{ for all } j\notin\Lambda^*\}$$

Clearly, there exists a one-to-one correspondence between configurations n_Λ and functions $J\in\mathscr{J}_\Lambda$. We thus can rewrite the partition function Z_Λ as in the following:

$$\begin{aligned}\sum_n e^{-\beta H_\Lambda(n)} &= \sum_n \prod_{j\in\Lambda^*} \left(\sum_{J_j\in\mathbf{Z}} \delta_{J_j,\, dn_j}\right) e^{-\beta H_\Lambda(n)} \\ &= \sum_n \prod_{j\in\Lambda^*} \left[\delta_{0,dn_j} + \sum_{J_j=1}^{\infty} (\delta_{J_j,\, dn_j} + \delta_{-J_j,\, dn_j})\right] e^{-\beta H_\Lambda(n)} \quad (2.1)\end{aligned}$$

Let $\nu>0$ and set $\xi_q = C_1 e^{-\nu|q|/2}$, where C_1 is a constant chosen so $\sum_{q=1}^{\infty}\xi_q = 1/2$. Then, replacing the coefficient of δ_{0,dn_j} for each $j\in\Lambda^*$ by $2\sum_{J_j=1}^{\infty}\xi_{J_j}$, the partition function (2.1) can be written in the following form[4–6]:

$$Z_\Lambda = \sum_{J\in\mathscr{J}_\Lambda^0} C_J Z^0(J) \qquad (2.2)$$

where $\mathscr{J}_\Lambda^0 = \{J\in\mathscr{J}_\Lambda : J_i \neq 0 \text{ for all } i\in\Lambda^*\}$, $C_J>0$ is such that $\sum_{J\in\mathscr{J}_\Lambda^0} C_J = 1$,

$$Z^0(J) = \sum_n \prod_{j\in\Lambda^*} [\delta_{dn_j,\,0} + \zeta_{J_j}(\delta_{dn_j,\, J_j} + \delta_{dn_j,\, -J_j})]\, e^{-\beta H_\Lambda^0(n)} \qquad (2.3)$$

where

$$\zeta_{J_j} = \tfrac{1}{2}\xi_{J_j}^{-1} e^{-\beta g(1) J_j^2} \qquad (2.4)$$

is the activity of the jump J_j at site $j\in\Lambda^*$, and the Hamiltonian H_Λ^0 is defined by

$$H_\Lambda(n) = g(1)\sum_{j\in\Lambda^*}(dn_j)^2 + H_\Lambda^0(n) \qquad (2.5)$$

We now introduce some notations. By $I(j,d)$ we denote the interval in $\mathbf{Z}$ (or $\mathbf{Z}^*$), centered at j with side d, i.e.,

$$I(j,d) = \left\{i\in\mathbf{Z}(\mathbf{Z}^*): |i-j| < \frac{d}{2}\right\}$$

Let Λ be a large interval centered at the origin, say $\Lambda = I(0, R)$. For a fixed $d_1 > 1$ we set $\Lambda_1 = \Lambda \cap d_1 \mathbf{Z}$, and similarly $\Lambda_1^* = \Lambda^* \cap d_1 \mathbf{Z}^*$; for $j \in \Lambda_1$ (or Λ_1^*) we let $I_1(j) = I(j, d_1)$.

Clearly,

$$\prod_{j \in \Lambda^*} [\delta_{dn_j, 0} + \zeta_{J_j}(\delta_{dn_j, J_j} + \delta_{dn_j, -J_j})]$$
$$= \prod_{j \in \Lambda_1^*} \left\{ \prod_{i \in I_1(j)} [\delta_{dn_i, 0} + \zeta_{J_i}(\delta_{dn_i, J_i} + \delta_{dn_i, -J_i})] \right\} \tag{2.6}$$

As in ref. 5, each term inside the curly bracket can be written as a convex combination of terms with the same form by using the following lemma:

Lemma 2.1. Let I be an index set with N elements and let $\zeta_j \geqslant 0$ and m_j, $J_j \in \mathbf{Z}$ be given for each $j \in I$. Then

$$\prod_{j \in I} [\delta_{m_j, 0} + \zeta_j(\delta_{m_j, J_j} + \delta_{m_j, -J_j})]$$
$$= \sum_{\sigma \in \mathscr{G}(I)} c_\sigma [\Delta_0(I, m) + \zeta_\sigma(\Delta_{J_\sigma}(I, m) + \Delta_{-J_\sigma}(I, m))]$$

where $\mathscr{G}(I) = \{\sigma: I \to \{0, 1, -1\}; \sigma \not\equiv 0\}$,

$$J_\sigma: \quad i \in I \to \mathbf{Z}$$
$$J_\sigma(i) = \sigma_i J_i$$
$$\Delta_J(I, m) = \prod_{i \in I} \delta_{m_i, J(i)}$$
$$\zeta_\sigma = \prod_{i \in I} [b_I \zeta_i]^{|\sigma_i|}$$

where b_I is given by $(1 + 2/b_I)^N = 3$, so

$$b_I \leqslant \frac{2}{\log 3} N$$

and $0 < c_\sigma$ is such that $\sum_{\sigma \in \mathscr{G}(I)} c_\sigma = 1$.

The proof of Lemma 2.1 is essentially done in Appendix A of ref. 5. Just replace in the above expansion the coefficient of $\prod_{j \in I} \delta_{m_j, 0}$ by $\sum_\sigma c_\sigma$ with $c_\sigma = (2 \prod_i b_I^{|\sigma_i|})^{-1}$.

We now need some definitions. A jump density is a function $J: D \to \mathbf{Z}$

with domain given by an arbitrary sequence of sites in $\mathbf{Z}^*$; we call i_D^+ (i_D^-) the largest (smallest) site of D and set $I_D = [i_D^-, i_D^+] \cap \mathbf{Z}^*$; J is said to be localized on the interval $I(j, d)$ if $D \subset \bar{I}(j, d) \equiv I(j, 3d)$.

A weighted jump density is a triple (J, ζ, D), where J is a jump density with domain $D \subset \mathbf{Z}^*$ and activity $\zeta \geqslant 0$. From now on all our jump densities will be weighted; we will write J for the triple (J, ζ, D) and will use $\zeta(J)$ and $D(J)$ for its corresponding activity and domain.

Thus, from (2.2), (2.3), and (2.6) and Lemma 2.1, the partition function Z_Λ can be written as a convex combination of partition functions of the type

$$\sum_n \prod_{j \in \Lambda_1^*} [\Delta_0(I_1(j), dn) + \zeta_j(\Delta_{J_j}(I_1(j), dn) + \Delta_{-J_j}(I_1(j), dn))] \, e^{-\beta H_\Lambda^1(n)} \tag{2.7}$$

where J_j is a weighted jump density localized on $I_1(j)$ with

$$\zeta_j \leqslant \prod_{\substack{i \in I_1(j): \\ J_j(i) \neq 0}} \left[\frac{2}{\log 3} d_1 \zeta_{J_j(i)}^{-1} e^{-\beta g(1) J_j(i)^2}\right] e^{-\beta h^1(I_1(j); n)} \tag{2.8}$$

where $h^1(D; n) = h(D; n) - \sum_{j \in D} g(1)\, dn_j$,

$$h(D; n) = \frac{1}{2} \sum_{k, l \in \hat{D}} g(k, l)(n_k - n_l)^2 \tag{2.9}$$

for any subset $D \subset \mathbf{Z}$ with $\hat{D} = \{j + 1/2\}_{j \in D} \cup \{j - 1/2\}_{j \in D}$. The Hamiltonian H_Λ^1 is given by

$$H_\Lambda^1(n) = H_\Lambda^0(n) - \sum_{j \in \Lambda_1^*} h(I_1(j); n)$$

Now, set

$$K_1 = K_1(\beta, g(1), d_1) = \frac{2}{\log 3} d_1 \sup_{q = 1, 2, \ldots} \sup_{\xi} \xi_q^{-1} e^{-\beta g(1) q^2/3} \tag{2.10}$$

We have that $\lim_{\beta \to \infty} K_1 = \lim_{g(1) \to \infty} K_1 = 0$ and if we pick β and g such that $K_1 < 1$, it follows from (2.8) and (2.10) that

$$\zeta_j \leqslant K_1 e^{-2\beta g(1) |J_j|/3} \tag{2.11}$$

where $|J_j| = \sum_{i \in I_1(j)} |J_j(i)|$.

We have proven the following theorem:

Theorem 2.2. Let $d_1 > 1$ be fixed. Then, if $K_1 < 1$, the partition function of the discrete Gaussian chain Z_Λ can always be written as a convex combination of partition functions of the form (2.7) with activities satisfying (2.11).

Remark. Theorem 2.2 can be trivially extended to include the external height partition function $Z_\Lambda(h)$ by just replacing (2.7) by

$$\sum_n \prod_{j \in \Lambda_1^*} [\Delta_0(I_1(j), dn) + \zeta_j(\Delta_{J_j}(I_1(j), dn) + \Delta_{-J_j}(I_1(j), dn))] \, e^{n(h)} e^{-\beta H_\Lambda^1(n)}$$

3. THE INDUCTIVE STEP

Let us fix $\alpha > 1$, the initial scale $d_1 = 3^{r_1}$, where $r_1 \in \{3, 4,...\}$, and $\Lambda = I(0, R)$. We define the successive scales by $d_{k+1} = 3^{r_{k+1}}$, where $r_{k+1} = [\alpha r_k]$ ($[t] = \sup\{r \in \mathbf{N}: r \leqslant t\}$) and set $d_0 = 1$.

We set $\Lambda_k = \Lambda \cap d_k \mathbf{Z}$, $I_k(j) = I(j, d_k)$ for $j \in \Lambda_k$ and $I_k^{k'}(j) = I_k(j) \cap d_{k'} \mathbf{Z}$ for $k' \leqslant k$. Notice that $\Lambda_0 = \Lambda$ and $\Lambda_N = \{0\}$, where $N \in \mathbf{N}$ is such that $d_{N-1} < R \leqslant d_N$.

We extend these definitions for the dual lattice $\mathbf{Z}^*$, which will be distinguished by an asterisk whenever necessary.

Definition. Let us fix a scale k, numbers δ, $\lambda > 0$, and $j \in \Lambda_k^*$. A weighted jump density $J = (J, \zeta, D)$ is (k, j, δ, λ)-admissible if

$$\begin{aligned} &\text{(i)} \quad D(J) \subset \bar{I}_k(j) \equiv I(j, 3d_k) \\ &\text{(ii)} \quad 0 \leqslant \zeta(J) \leqslant d_k^{-\delta} e^{-(\lambda + 1/\log d_k)|J|} \end{aligned} \tag{3.1}$$

where $|J| = \sum_{j \in D(J)} |J(j)|$ (we allow $J \equiv 0$ with $D(J) \neq \varnothing$, but we require $\zeta(J) = 0$).

A jump density J is said to be neutral if $Q_J \equiv \sum_{j \in D(J)} J(j) = 0$.

Definition. Let $p > 2$ be fixed, $k \in \mathbf{N}$, $j \in \Lambda_k$, and δ, $\lambda > 0$. A collection $\mathcal{N}_{(k,j,\delta,\lambda)}$ of neutral jump densities will be called a (k, j, δ, λ)-sparse neutral ensemble if:

(i) For $k = 1$,

$$\mathcal{N}_{(1,j,\delta,\lambda)} = \varnothing$$

(ii) For $k = 2, 3,...$ we have

$$\mathcal{N}_{(k,j,\delta,\lambda)} = \left[\bigcup_{i \in I_k^{k-1}(j)} \mathcal{N}_{(k-1,i,\delta,\lambda)} \right] \cup \{(J, \zeta, D)\}$$

where each $\mathcal{N}_{(k-1,i,\delta,\lambda)}$ is a $(k-1, i, \delta, \lambda)$-sparse neutral ensemble, (J, ζ, D) is a $(k-1, i', \delta, \lambda)$-admissible neutral jump density for some $i' \in I_k^{k-1}(j)$ such that $I(i', \frac{1}{3}d_k) \subset \bar{I}_k(i)$ with (3.1) replaced by

$$\zeta \leqslant 3\left(\frac{2}{\log 3}\right)^2 d_{k-1}^{-\delta} e^{-\lambda |J|}$$

and

$$2 \leqslant |J| \leqslant (\log d_{k-1})^p$$

Given $\mathcal{N}_{(k,j,\delta,\lambda)}$, let

$$\Gamma(\mathcal{N}_{(k,j,\delta,\lambda)}; n) = \prod_{J \in \mathcal{N}_{(k,j,\delta,\lambda)}} [\Delta_0(D(J); n) + \zeta(J)(\Delta_J(D(J); n) + \Delta_{-J}(D(J); n))] \tag{3.2}$$

where

$$\Delta_T(D(J); n) = \prod_{j \in D(J)} \delta_{dn_j, T(j)}$$

with $T = 0, J, -J$.

Definition. A collection $\mathcal{F}$ of weighted jump densities is said to be compatible with the interval $I \subset \mathbf{Z}^*$ iff:

(i) $D(J) \cap D(J') = \varnothing$ for all $J, J' \in \mathcal{F}$ with $J \neq J'$.

(ii) $\bigcup_{J \in \mathcal{F}} D(J) = I$.

Definition. Given a scale k, a (k, δ, λ)-regular jump assignment $\mathcal{A}_{(k,\delta,\lambda)}$ is a collection of weighted jump densities compatible with Λ^* given by

$$\{\mathcal{N}_{(k,j,\delta,\lambda)}, (J_j, \zeta_j, D_j)\}_{j \in \Lambda_k^*}$$

where each $\mathcal{N}_{(k,j,\delta,\lambda)}$ is a (k, j, δ, λ)-sparse neutral ensemble and each (J_j, ζ_j, D_j) is a $(k, j, \delta + \alpha - 1, \lambda)$-admissible jump density.

Definition. A (k, δ, λ)-regular partition function is a partition function of the form

$$Z_{(k,\delta,\lambda)} = \sum_n \prod_{j \in \Lambda_k^*} [\Gamma(\mathcal{N}_{(k,j,\delta,\lambda)}; n)\, \gamma(J_j; n)]\, e^{-\beta H_\Lambda^k(n)} \tag{3.3}$$

where $\mathscr{A}_{(k,\delta,\lambda)} = \{\mathscr{N}_{(k,j,\delta,\lambda)}, J_j\}_{j\in \Lambda_k^*}$ is a (k, δ, λ)-regular jump assignment,

$$\gamma(J; n) = \Delta_0(D(J); n) + \zeta(J)(\Delta_J(D(J); n) + \Delta_{-J}(D(J); n))$$

and $H_\Lambda^k(n) = H_\Lambda^k(\mathscr{A}_{(k,\delta,\lambda)}; n)$ is given by

$$H_\Lambda^k(n) = H_\Lambda(n) - \sum_{J\in \mathscr{A}_{(k,\delta,\lambda)}} h(D(J); n)$$

with $h(D; n)$ the self-energy of J, as defined in (2.9).

In this language, Theorem 2.2 states that, for a choice of parameters such that $K_1(\beta, g(1), d_1) \leqslant d_1^{-\delta-\alpha+1}$ and $2\beta g(1)/3 - 1/\log d_1 \geqslant \lambda$, the partition function Z_Λ given by (1.4) can be written as a convex combination of $(1, \delta, \lambda)$-regular partition functions. Theorem 2.2 gives the initial step in the inductive procedure of the following theorem:

Theorem 3.1. Let $1 < \alpha < 2$ and

$$\frac{\alpha(\alpha-1)}{2-\alpha} < \delta < \beta(1-a) - \alpha$$

with $a = a(\alpha, d_1)$ such that $\lim_{d_1\to\infty} a = 0$. Suppose $\lambda \leqslant 2\beta g(1)/3 - 1/\log d_1$ and $K_1 \leqslant d_1^{-\delta-\alpha+1}$. Then, if d_1 is sufficiently large, the partition function of a discrete Gaussian chain Z_Λ can always be written as a convex combination of (k, δ, λ)-regular partition functions for any $k = 1, 2,..., N$.

Theorem 3.1 follows from Theorem 2.2 and from the following result:

Lemma 3.2. Let α, δ, λ, d_1 be as above. Then if d_1 is sufficiently large, any (k, δ, λ)-regular partition function can be written as a convex combination of $(k+1, \delta, \lambda)$-regular partition functions.

Remarks. (1) Theorem 3.1 and Lemma 3.2 may include the external height partition function (1.6) by simply adding the term $e^{n(h)}$ in expression (3.3).

(2) Theorem 3.1 and Lemma 3.2 may also be applied to the partition function Z_Λ of the $1/(i-j)^2$ Ising chain. In this case, we let n be in the set of configurations $\{n_j \in \{1, -1\}\}_{j\in \mathbf{Z}}$ with $n_j = 1$ for $j \notin \Lambda$. As the boundary condition breaks the symmetry $n \to -n$ we need to replace, for all $J \in \{\mathscr{N}_{(k,j,\delta,\lambda)}, J_j\}_{j\in \Lambda_{k+1}^*}$, $\Delta_0(D(J); n) + \zeta(J)(\Delta_J(D(J); n) + \Delta_{-J}(D(J); n))$ in (3.3) by its asymmetric version

$$\Delta_0(D(J); n) + \zeta(J)\, \Delta_J(D(J); n)$$

where $J: D(J) \to \{0, 1\}$ is the Ising weighted flip density defined by $dn_j \equiv \frac{1}{2} |n_{j+1/2} - n_{j-1/2}| \in \{0, 1\}$.

Proof of Lemma 3.2. Let $k \in \{1, 2, ..., N-1\}$ and let $\{\mathcal{N}_{(k,j,\delta,\lambda)}, J_j\}_{j \in \Lambda_k^*}$ be a (k, δ, λ)-regular assignment. Let $Z_{(k,\delta,\lambda)}$ be given by (3.3). We have

$$Z_{(k,\delta,\lambda)} = \sum_n \prod_{j \in \Lambda_{k+1}^*} \left\{ \Gamma(\mathcal{N}^{\#}_{(k+1,j,\delta,\lambda)}; n) \prod_{i \in I_{k+1}^k(j)} \gamma(J_i) \right\} e^{-\beta H_\Lambda^k(n)} \tag{3.4}$$

where, for each $j \in \Lambda_{k+1}^*$,

$$\mathcal{N}^{\#}_{(k+1,j,\delta,\lambda)} = \bigcup_{i \in I_{k+1}^k(j)} \mathcal{N}_{(k,i,\delta,\lambda)}$$

is, by definition, a $(k+1, j, \delta, \lambda)$-sparse neutral ensemble.

Using Lemma 2.1, we can write (3.4) as a convex combination of partition functions of the form

$$\sum_n \prod_{j \in \Lambda_{k+1}^*} [\Gamma(\mathcal{N}^{\#}_{(k+1,j,\delta,\lambda)}, n)\, \gamma(J_j^{\#})] \exp[-\beta H_\Lambda^{\# k+1}(n)] \tag{3.5}$$

where each $J_j^{\#}$ is of the form

$$J_j^{\#} = \sum_{i \in I_{k+1}^k(j)} \sigma_i J_i$$

for some $\sigma \in \mathcal{G}(I_{k+1}^k(j))$,

$$\zeta_j^{\#} \leqslant \prod_{i \in I_{k+1}^k(j)} \left[\frac{2}{\log 3} d_k^{-\delta}\right]^{|\sigma_i|} \exp\left[-\left(\lambda + \frac{1}{\log d_k}\right) |J_j^{\#}|\right] \exp\,[-\beta h^{k+1}(J_j^{\#}; n)] \tag{3.6}$$

where

$$h^{k+1}(J_j^{\#}; n) = h(D(J_j^{\#}); n) - \sum_{i \in I_{k+1}^k(j)} h(D(J_i); n) \tag{3.7}$$

and

$$D_j^{\#} = \bigcup_{i \in I_{k+1}^k(j)} D_i$$

Moreover, the collection $\mathcal{A}_{k+1}^{\#} = \{\mathcal{N}^{\#}_{(k+1,j,\delta,\lambda)}, J_j^{\#}\}_{j \in \Lambda_{k+1}^*}$ is compatible with Λ^*, i.e.,

$$\bigcup_{j \in \Lambda_{k+1}^*} \left[D_j^{\#} \bigcup_{J \in \mathcal{N}^{\#}_{(k+1,j,\delta,\lambda)}} D(J) \right] = \Lambda^*$$

and $H_\Lambda^{\#^{k+1}}(n)$ is given by

$$H_\Lambda^{\#^{k+1}}(n) = H_\Lambda(n) - \sum_{J \in \mathscr{A}_{k+1}^{\#}} h(D^{\#}(J); n)$$

To propagate our bound on the activities $\zeta_j^{\#}$, we will need to estimate in some cases $h^{k+1}(J_j^{\#}; n)$. As in ref. 5, this will be done by the following lemma:

Lemma 3.3. Let $\mathscr{A}_{k+1}^{\#}$ be as above. Suppose we have for some $j_0 \in \Lambda_{k+1}^{*}$:

(a_1) $J_{j_0}^{\#} = J_{i_0}$ for $i_0 \in I_{k+1}^{k}(j_0)$ with $Q_{J_{j_0}^{\#}} = q$.

(a_2) $I(i_0, 1/3 d_{k+1}) \cap D_j^{\#} = \varnothing$ for all $j \in \Lambda_{k+1}^{*}$ with $j \neq j_0$.

Then, for any $\mathscr{B} \subset \mathscr{A}_{k+1}^{\#}$ such that $J_{j_0}^{\#} \in \mathscr{B}$

$$h^{k+1}(J_{j_0}^{\#}; n^{\mathscr{B}}) \geqslant q^2(1-a) \log \frac{d_{k+1}}{d_k}$$

where $n^{\mathscr{B}} = \sum_{J \in \mathscr{B}} n^J$ is the configuration determined by the ensemble $\mathscr{B}$, n^J is defined by

$$n_i^J = \begin{cases} \sum_{\substack{l \in D(J): \\ l \leqslant i}} J(l) & \text{if} \quad i \in I_{D(J)} \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

and $a = a(d_1, \alpha)$ is a positive constant such that $\lim_{d_1 \to \infty} a = 0$.

Lemma 3.3 will be proven in Appendix A.

Lemma 3.3 requires (a_2), which may not be true. Notice that if there exist a $J_j^{\#}$ which does not satisfy (a_2), it must be either the right or the left neighbor of $J_{j_0}^{\#}$. It could also happen that $J_j^{\#}$ violates (a_2) with respect to both neighbors, $J_{j_0}^{\#}$ and $J_{j_0'}^{\#}$, satisfying (a_1).

Let us define the equivalence

$$j \sim j' \Leftrightarrow \begin{cases} J_j^{\#} \text{ satisfies } (a_1);\ J_{j'}^{\#} \text{ does not satisfy } (a_2) \\ \text{or} \\ J_{j'}^{\#} \text{ satisfies } (a_1);\ J_j^{\#} \text{ does not satisfy } (a_2) \end{cases}$$

[we allow $j = j'$, so $j \sim j$; we set $j \sim j'$ if $J_j^{\#}$, $J_{j'}^{\#}$ satisfy (a_1) with $j \sim j''$, $j' \sim j''$ for some j''], and let $Y_1,..., Y_s$ denote the distinct equivalent classes. Notice that we always have $|Y_i| = 1, 2$, or 3.

For each Y_i we apply Lemma 2.1 for the $\gamma(J_j^{\#})$ with $j \in Y_i$ and define

$$J_{Y_i}^{\#} = \sum_{j \in Y_i} \tau_j J_j^{\#}$$

where $\tau_j = 0, \pm 1$ with $\sum_{j \in Y_i} |\tau_j| \neq 0$, and

$$D_{Y_i}^{\#} = \bigcup_{j \in Y_i} D_j^{\#}$$

We then have that (3.5) can be written as a convex combination of the same type, but with (a$_1$) holding, given by

$$\sum_n \prod_{j \in \Lambda_{k+1}^*} [\Gamma(\tilde{\mathcal{N}}_{(k+1, j, \delta, \lambda)}; n)\, \gamma(\tilde{J}_j)]\, e^{-\beta H_\Lambda^{k+1}(n)}$$

where each $\tilde{J}_j$ is either identically 0 (and in this case $\tilde{D}_j = \varnothing$) or

$$\tilde{J}_j = J_{Y_i}^{\#}$$

for some $i = 1, 2,..., s$, with

$$\tilde{\zeta}_j = \zeta_{Y_i}^{\#} \leqslant \prod_{l \in Y_i} \left[\frac{6}{\log 2} \zeta_l^{\#} \right]^{|\tau_l|} \exp[-\beta h^{k+1}(J_{Y_i}^{\#}; n)] \tag{3.9}$$

where

$$h^{k+1}(J_{Y_i}^{\#}; n) = h(D_{Y_i}^{\#}; n) - \sum_{j \in Y_i} h(D_j^{\#}; n)$$

$\tilde{D}_j = D_{Y_i}^{\#} \subset \bar{I}_{k+1}(j)$ is such that

$$\bigcup_{j \in \Lambda_{k+1}^*} \left[\tilde{D}_j \bigcup_{J \in \tilde{\mathcal{N}}_{(k+1, j, \delta, \lambda)}} D(J) \right] = \Lambda^*$$

where $\tilde{\mathcal{N}}_{(k+1, j, \delta, \lambda)} = \mathcal{N}_{(k+1, j, \delta, \lambda)}^{\#}$ and

$$H_\Lambda^{k+1}(n) = H_\Lambda(n) - \sum_{J \in \tilde{\mathscr{A}}_{k+1}} h(D(J); n)$$

with $\tilde{\mathscr{A}}_{k+1} = \{\tilde{\mathcal{N}}_{(k+1, j, \delta, \lambda)}; \tilde{J}_j\}_{j \in \Lambda_{k+1}^*}$.

We now return to the estimate of $\tilde{\zeta}_j$.

Given $j \in \Lambda_{k+1}^*$, let N_j^k be the number of components of $\tilde{J}_j$ on the previous scale k, i.e.,

$$N_j^k = \sum_{i = j, j \pm 1} |\tau_i| \sum_{l \in I_{k+1}^k(i)} |\sigma_l|$$

We consider several cases:

(i) $N_j^k \geqslant 2$. In this case we define $\mathscr{N}_{(k+1,j,\delta,\lambda)} = \tilde{\mathscr{N}}_{(k+1,j,\delta,\lambda)}$ and notice that (3.6) and (3.9) give us

$$\tilde{\zeta}_j \leqslant \left(\frac{6}{\log 3}\right)\left(\frac{2}{\log 3}\right)^2 d_k^{-2\delta} \exp\left[-\left(\lambda + \frac{1}{\log d_k}\right)|\tilde{J}_j|\right]$$
$$\leqslant d_{k+1}^{-\delta-\alpha+1} \exp\left[-\left(\lambda + \frac{1}{\log d_{k+1}}\right)|\tilde{J}_j|\right]$$

if $\delta > \alpha(\alpha-1)/(2-\alpha)$ and d_1 is sufficiently large. We define $J_j = \tilde{J}_j$.

(ii) $N_j^k = 1$. Here we consider three subcases:

(iia) $|\tilde{J}_j| \geqslant (\log d_k)^p$. We let $\mathscr{N}_{(k+1,j,\delta,\lambda)} = \tilde{\mathscr{N}}_{(k+1,j,\delta,\lambda)}$ and $J_j = \tilde{J}_j$. Then (3.1) follows for J_j in the $(k+1)$th scale from (3.6) and (3.9), since if d_1 is sufficiently large, we have

$$(\log d_1)^{p-2} > \alpha(\delta + \alpha)$$

(iib) $|\tilde{J}_j| < (\log d_k)^p$ and $Q_{\tilde{J}_j} = 0$. We decompose $\tilde{J}_j$ into two weighted jump densities, $\tilde{J}_j^1$ and $\tilde{J}_j^2$, where $(\tilde{J}_j^1, \tilde{\zeta}_j^1, \tilde{D}_j^1)$ is defined by

$$\tilde{J}_j^1(i) = \tilde{J}_j(i) \qquad \text{for} \quad i \in \tilde{D}_j^1 = \bar{I}_k(i') \cap \tilde{D}_j$$

with i' such that $\operatorname{supp} \tilde{J}_j \subset \bar{I}_k(i')$ and

$$\tilde{\zeta}_j^1 = \tilde{\zeta}_j$$

Then $\tilde{J}_j^2 = (\tilde{J}_j^2, \tilde{\zeta}_j^2, \tilde{D}_j^2)$ is given by

$$\tilde{J}_j^2 \equiv 0, \qquad \tilde{\zeta}_j^2 = 0, \qquad \text{and} \qquad \tilde{D}_j^2 \cap \tilde{D}_j^1 = \varnothing$$

with $\tilde{D}_j^1 \cup \tilde{D}_j^2 = \tilde{D}_j$. We let $J_j = \tilde{J}_j^2$ and

$$\mathscr{N}_{(k+1,j,\delta,\lambda)} = \tilde{\mathscr{N}}_{(k+1,j,\delta,\lambda)} \cup \{\tilde{J}_j^1\}$$

and notice that the latter is a $(k+1, j, \delta, \lambda)$-sparse neutral ensemble.

(iic) $|\tilde{J}_j| < (\log d_k)^p$ and $Q_{\tilde{J}_j} \neq 0$. We define $\mathscr{N}_{(k+1,j,\delta,\lambda)} = \tilde{\mathscr{N}}_{(k+1,j,\delta,\lambda)}$ and use Lemma 3.3 to obtain

$$\tilde{\zeta}_j \leqslant 3\left(\frac{2}{\log 3}\right)^2 d_k^{-\delta}\left(\frac{d_{k+1}}{d_k}\right)^{-\beta(1-a)} \exp\left[-\left(\lambda + \frac{1}{\log d_k}\right)|\tilde{J}_j|\right]$$
$$\leqslant d_{k+1}^{-\delta-\alpha+1} \exp\left[-\left(\lambda + \frac{1}{\log d_{k+1}}\right)|\tilde{J}_j|\right]$$

if $\delta < \beta(1-a) - \alpha$ and d_1 is sufficiently large.

This concludes the proof of Lemma 3.2 and Theorem 3.1.

4. PROOF OF THEOREM 1.1: PEIERLS ARGUMENT

We now will show how to use Theorem 3.1 to estimate expectations and prove Theorem 1.1.

For any $\beta > 1$ we pick $1 < \alpha < 2$, d_1 sufficiently large, and positive numbers λ, δ such that

$$\frac{\beta}{2} g(1) \leqslant \lambda \leqslant \frac{2\beta}{3} g(1) - \frac{1}{\log d_1}$$

and

$$\frac{\alpha(\alpha-1)}{2-\alpha} < \delta < \beta(1-a) - \alpha \tag{4.1}$$

Notice that (4.1) requires $\beta > (1-a)^{-1} \alpha/(2-\alpha)$, which can always be satisfied for $\beta > 1$ by picking α sufficiently close to 1 and d_1 sufficiently large, since $\lim_{d_1 \to \infty} a = 0$.

Let $\bar{g}$ be given by

$$K_1(1, \bar{g}, d_1) = d_1^{-\delta - \alpha + 1} \tag{4.2}$$

From (2.10) we have that for any $g(1) \geqslant \bar{g}$ and $\beta > 1$,

$$K_1(\beta, g(1), d_1) < K_1(1, \bar{g}, d_1)$$

and Theorem 3.2 asserts that the external height partition function can be written as

$$Z_\Lambda(h) = \sum_\gamma c_\gamma Z^\gamma_{(N,\delta,\lambda)}(h)$$

where $c_\gamma > 0$ is such that $\sum_\gamma c_\gamma = 1$ and for each γ,

$$Z^\gamma_{(N,\delta,\lambda)}(h) = \sum_n e^{n(h)} \Gamma(\mathcal{N}^\gamma_{(N,0,\delta,\lambda)}; n) \tag{4.3}$$

is an (N, δ, λ)-regular external height partition function.

Hence, the external height expectation (1.5) can be written as

$$\begin{aligned} \langle e^{n(h)} \rangle_\Lambda &= \frac{\sum_\gamma c_\gamma Z^\gamma_{(N,\delta,\lambda)}(h)}{\sum_\gamma c_\gamma Z^\gamma_{(N,\delta,\lambda)}(0)} \\ &= \sum_\gamma d_\gamma \frac{Z^\gamma_{(N,\delta,\lambda)}(h)}{Z^\gamma_{(N,\delta,\lambda)}(0)} \end{aligned} \tag{4.4}$$

where d_γ is such that $\sum_\gamma d_\gamma = 1$.

Notice that since $\zeta(J) \geqslant 0$ for any $J \in \mathcal{N}^{\gamma}_{(N,0,\delta,\lambda)}$, it follows that $Z^{\gamma}_{(N,\delta,\lambda)}(0) \geqslant 1 > 0$ and (4.4) is well defined.

Let us fix γ and set $\mathcal{N} \equiv \mathcal{N}^{\gamma}_{(N,0,\delta,\lambda)}$ and $Z_{\mathcal{N}}(h) \equiv Z^{\gamma}_{(N,\delta,\lambda)}(h)$. Since the only n-configurations involved in the partition function $Z_{\mathcal{N}}$ are those determined by the J's in $\mathcal{N}$, (4.3) can be expanded to get

$$Z_{\mathcal{N}}(h) = \sum_{\sigma \in \mathcal{P}_{\mathcal{N}}} \zeta_\sigma \exp[n^{J_\sigma}(h)] \exp[-\beta H(\mathcal{N}; n^{J_\sigma})] \tag{4.5}$$

where $\mathcal{P}_{\mathcal{N}} = \{\sigma: J \in \mathcal{N} \to \sigma_J \in \{0, 1, -1\}\}$,

$$J_\sigma = \sum_{J \in \mathcal{N}} \sigma_J J$$

$$\zeta_\sigma = \prod_{J \in \mathcal{N}} [\zeta(J)]^{|\sigma_J|}$$

$$n^{J_\sigma}(h) = \sum_{k \in \mathbf{Z}} n_k^{J_\sigma} h_k$$

where $n_j^{J_\sigma}$ is given by (3.8), with

$$D(J_\sigma) = \bigcup_{\substack{J \in \mathcal{N}: \\ \sigma_J \neq 0}} D(J)$$

and

$$H(\mathcal{N}; n) = H_\Lambda(n) - \sum_{J \in \mathcal{N}} h(D(J); n)$$

As $n_j^{J_\sigma} = -n_j^{J_{-\sigma}}$, $\zeta_\sigma = \zeta_{-\sigma}$, and $H(\mathcal{N}; n) = H(\mathcal{N}; -n)$, (4.5) can be written as

$$Z_{\mathcal{N}}(h) = \sum_{\sigma \in \mathcal{P}_{\mathcal{N}}} \zeta_\sigma \cosh n^{J_\sigma}(h) \exp[-\beta H(\mathcal{N}; n^{J_\sigma})] \tag{4.6}$$

Given J_σ, we now need to estimate $n_j^{J_\sigma}$. Notice that $n_j^{J_\sigma}$ gives the height of the jump density J_σ at $j \in \Lambda$ and the neutrality of $J \in \mathcal{N}$ implies that the height $n_j^{J_\sigma}$ depends only on the variables $J \in \mathcal{X}_j$ defined by

$$\mathcal{X}_j = \{J \in \mathcal{N}: i^-_{D(J)} < j < i^+_{D(J)}\}$$

In particular,

$$n_j^{J_\sigma} = 0 \qquad \text{if} \quad \sigma \in \mathcal{P}_{\mathcal{N}/\mathcal{X}_j} \tag{4.7}$$

Let us fix $\sigma \in \mathscr{P}_{\mathscr{N}}$ with $\sigma_{\tilde{J}} \neq 0$ for some $\tilde{J} \in \mathscr{X}_j$ and define $\tilde{\sigma} \in \mathscr{P}_{\mathscr{N}}$ by

$$\tilde{\sigma}_J = \begin{cases} 0 & \text{if} \quad J = \tilde{J} \\ \sigma_J & \text{otherwise} \end{cases}$$

We then have

$$n_j^{J_\sigma} \leqslant n_j^{J_{\tilde{\sigma}}} + \frac{|\tilde{J}|}{2}$$

Iterating this expression and using (4.7), we obtain

$$n_j^{J_\sigma} \leqslant \frac{1}{2} \sum_{J \in \mathscr{X}_j} |\sigma_J| \, |J| \tag{4.8}$$

We now are ready to .prove Theorem 1.1. Let h be the one-point external height density with $x_0 = 0$. From (4.6)–(4.8) we have

$$|Z_{\mathscr{N}}(h)| \leqslant \sum_{\sigma \in \mathscr{P}_{\mathscr{X}_0}} \zeta_\sigma \cosh \frac{h^0}{2} |J_\sigma| \sum_{\sigma' \in \mathscr{P}_{\mathscr{N}/\mathscr{X}_0}} \zeta_{\sigma'} \exp[-\beta H(\mathscr{N}; n^{J_\sigma} + n^{J_{\sigma'}})] \tag{4.9}$$

As $\zeta(J) \geqslant 0$ we have

$$Z_{\mathscr{N}}(0) = \sum_{\sigma' \in \mathscr{P}_{\mathscr{N}}} \zeta_{\sigma'} e^{-\beta H(\mathscr{N}; n^{J_{\sigma'}})} \geqslant \sum_{\sigma' \in \mathscr{P}_{\mathscr{N}/\mathscr{X}_0}} \zeta_{\sigma'} e^{-\beta H(\mathscr{N}; n^{J_{\sigma'}})} \tag{4.10}$$

To complete our Peierls argument, we will need to perform a cancellation in each term of (4.4). This will be done by using the following lemma:

Lemma 4.1. Let $\sigma \in \mathscr{P}_{\mathscr{X}_0}$, $\sigma' \in \mathscr{P}_{\mathscr{N}/\mathscr{X}_0}$. Then, if d_1 is sufficiently large,

$$\Delta H(\mathscr{N}; \sigma, \sigma') = H(\mathscr{N}; n^{J_\sigma} + n^{J_{\sigma'}}) - H(\mathscr{N}; n^{J_{\sigma'}}) \geqslant 0$$

Lemma 4.1 will be proved in Appendix B.

In view of (4.9), (4.10), and Lemma 4.1, it follows that

$$\begin{aligned} \frac{|Z_{\mathscr{N}}(h)|}{Z_{\mathscr{N}}(0)} &\leqslant \sum_{\sigma \in \mathscr{P}_{\mathscr{X}_0}} \zeta_\sigma \cosh \frac{h^0}{2} |J_\sigma| \\ &\leqslant \prod_{J \in \mathscr{X}_0} \left[1 + 2\zeta(J) \cosh \frac{h^0}{2} |J| \right] \\ &\leqslant \prod_{k=1}^{N} \left[1 + 6 \left(\frac{2}{\log 3} \right)^2 d_k^{-\delta} e^{-2\lambda} \cosh h^0 \right] \\ &\leqslant \exp(\theta e^{-2\lambda} \cosh h^0) \end{aligned}$$

provided $|h^0| < 2\lambda$, where $\theta = \theta(d_1, \alpha)$ is a positive constant independent of γ and Λ (recall that $|J| \geqslant 2$ for any $J \in \mathcal{N}$).

This finishes the proof of Theorem 1.1(a).

Given $i, j, k \in \Lambda$ such that $i \neq j$, we now let $\mathcal{X}_{ij} \subset \mathcal{N}$ be given by

$$\mathcal{X}_{ij} = \{J \in \mathcal{N} : i^-_{D(J)} < i, j < i^+_{D(J)}\}$$

and set

$$\mathcal{X}^{ij}_k = \{J \in \mathcal{X}_k : J \notin \mathcal{X}_{ij}\}$$

Given $1 < N_0 < N$, let h be the two-point density with $x_0 = 0$, $y_0 = x$ such that $d_{N_0} < |x| \leqslant d_{N_0+1}$. As in (4.9), the external height partition function $Z_{\mathcal{N}}(h)$ can be bounded by

$$|Z_{\mathcal{N}}(h)| \leqslant \prod_{l=0,x} \left(\sum_{\sigma \in \mathcal{P}_{\mathcal{X}_l^{0x}}} \zeta_\sigma \cosh \frac{h^0}{2} |J_\sigma| \right) \sum_{\sigma' \in \mathcal{P}_{\mathcal{X}_{0x}}} \zeta_{\sigma'} \cosh h^0 \, |J_{\sigma'}|$$

$$\times \sum_{\omega \in \mathcal{P}_{\mathcal{N}/\mathcal{X}_0 \cup \mathcal{X}_x}} \zeta_\omega \exp[-\beta H(\mathcal{N}; n^{J_\sigma + J_{\sigma'}} + n^{J_\omega})]$$

(recall that $\mathcal{X}_{ij} \cup \mathcal{X}^{ij}_i \cup \mathcal{X}^{ij}_j = \mathcal{X}_i \cup \mathcal{X}_j$).

Hence, if $|h^0| < 2\lambda$, by using Lemma 4.1, we have

$$\frac{|Z_{\mathcal{N}}(h)|}{Z_{\mathcal{N}}(0)} \leqslant \prod_{l=0,x} \left\{ \prod_{J \in \mathcal{X}_l^{0x}} \left[1 + 2\zeta(J) \cosh \frac{h^0}{2} |J| \right] \right\}$$

$$\times \prod_{J' \in \mathcal{X}_{0x}} [1 + 2\zeta(J') \cosh h^0 \, |J'|]$$

$$\leqslant [\exp(\theta e^{-2\lambda} \cosh h^0)]^2 \prod_{k=N_0+1}^{N} \left[1 + \left(\frac{4}{\log 3} \right)^2 d_k^{-\delta} e^{-2\lambda} \cosh h^0 \right]$$

Thus, (1.12) follows and Theorem 1.1 (b) is proved.

We now will show how Theorem 3.1 and Lemma 3.2 can be used to prove spontaneous magnetization in the one-dimensional Ising model with $1/(i-j)^2$ interaction. From Remark 2 of Section 3 and (4.1)–(4.5), the expectation $\langle 1 - n_0 \rangle$ can be written as a convex combination of expectations of the form

$$\sum_{\sigma \in \mathcal{P}_{\mathcal{N}}} \zeta_\sigma (1 - n_0^{J_\sigma}) \exp[-\beta H(\mathcal{N}; n^{J_\sigma})] \Big/ \sum_{\sigma' \in \mathcal{P}_{\mathcal{N}}} \zeta_{\sigma'} \exp[-\beta H(\mathcal{N}; n^{J_{\sigma'}})] \tag{4.11}$$

where $\mathscr{P}_{\mathscr{N}} = \{\sigma: J \in \mathscr{N} \to \{0, 1\}\}$, J_σ and ζ_σ are given as in (4.5), and $n_j^J \in \{1, -1\}$ is the spin value at $j \in \Lambda$ in the flip density J, given by

$$n_j^J = \begin{cases} \prod_{\substack{i \in D(J): \\ i \leqslant j}} (-1)^{J(i)} & \text{if } j \in I_{D(J)} \\ 1 & \text{otherwise} \end{cases}$$

Given $j \in \Lambda$, let

$$\mathscr{P}^j_{\mathscr{N}} = \{\sigma \in \mathscr{P}_{\mathscr{N}} : n_j^{J_\sigma} = -1\}$$

Clearly, for each $\sigma \in \mathscr{P}^j_{\mathscr{N}}$ there exists at least one $\omega = \omega(\sigma) \in \mathscr{P}_{\mathscr{N}}$ satisfying:

(i) For some $\tilde{J} \in \mathscr{N}$ with $\sigma_{\tilde{J}} = 1$,

$$\omega_J = \begin{cases} 0 & \text{if } J = \tilde{J} \\ \sigma_J & \text{otherwise} \end{cases}$$

(ii) $n_j^{J_\omega} = 1$.

We let

$$\tilde{\mathscr{P}}^j_{\mathscr{N}} = \{\omega(\sigma)\}_{\sigma \in \mathscr{P}^j_{\mathscr{N}}}$$

with $\omega(\sigma)$ being as above.

We use $\zeta_\sigma \geqslant 0$ and Lemma 4.1 to obtain that (4.11) can be bounded by

$$\begin{aligned} &\leqslant 2 \sum_{\sigma \in \mathscr{P}^{x_0}_{\mathscr{N}}} \zeta_\sigma \exp[-\beta H(\mathscr{N}; n^{J_\sigma})] \Big/ \sum_{\omega \in \tilde{\mathscr{P}}^{x_0}_{\mathscr{N}}} \zeta_\omega \exp[-\beta H(\mathscr{N}; n^{J_\omega})] \\ &\leqslant 2 \sup_{\tilde{J} \in \mathscr{N}} \zeta(\tilde{J}) \\ &\leqslant 6 \left(\frac{2}{\log 3}\right)^2 d_1^{-\delta} e^{-\beta g(1)} < 1 \end{aligned}$$

if $g(1)$ is sufficiently large.

APPENDIX A

Proof of Lemma 3.3. Given $j_0 \in \Lambda^*_{k+1}$, let $\hat{J} = J^{\#}_{j_0}$ as in (a_1), $\hat{D} = D(J^{\#}_{j_0})$, $\hat{I}_l = I_{\hat{D}} \cap \Lambda^*_l$, for $l \leqslant k$,

$$\hat{\mathscr{N}} = \bigcup_{j \in \hat{I}_k} \mathscr{N}_{(k, j, \delta, \lambda)}$$

Clearly the collection $\{\hat{D}, \{D(J)\}_{J\in\hat{\mathcal{N}}}\}$ has disjoint support and

$$\hat{D} \bigcup_{J\in\hat{\mathcal{N}}} D(J) = I_{\hat{D}} \tag{A.1}$$

(see definitions in Section 3).

Let $i_0 \in I^k_{k+1}(j_0)$ as in assumption (a_1) and set

$$\Sigma = I_{\hat{D}}/I(i_0, 3d_k) = \Sigma^- \cup \Sigma^+$$

where $\Sigma^{\pm}$ is the connect interval of Σ at the right (left) of i_0, i.e.,

$$\Sigma^{\pm} = \{j \in \Sigma\colon j > i_0 \ (j < i_0)\}$$

We also need to introduce

$$\hat{\Sigma} = \left\{\bigcup_{J\in\hat{\mathcal{N}}} D(J)\right\} \cap \Sigma = \hat{\Sigma}^- \cup \hat{\Sigma}^+$$

where $\hat{\Sigma}^{\pm}$ is defined as above. Notice that due (A.1), we have

$$\Sigma/\hat{\Sigma} \subset \hat{D} \tag{A.2}$$

Now let $\mathscr{B}$ be as in the assumptions of Lemma 3.3. By neutrality of $J \in \hat{N}$, we have

$$h^{k+1}(\hat{J}, n^{\mathscr{B}}) = h^{k+1}(\hat{J}, n^{\hat{J}}) \tag{A.3}$$

[see definitions (3.7) and (2.9)].

From (A.2) and assumption (a_1), (A.3) can be bounded by

$$\geqslant q^2 \sum_{i,j} \chi_{\Sigma^-/\hat{\Sigma}^-}(i)\, g(i-j)\, \chi_{\Sigma^+/\hat{\Sigma}^+}(j)$$

$$\geqslant q^2 \sum_{i,j} \{\chi_{\Sigma^-}(i)\, \chi_{\Sigma^+}(j) - [\chi_{\Sigma^-}(i)\, \chi_{\hat{\Sigma}^+}(j) + \chi_{\hat{\Sigma}^-}(i)\, \chi_{\Sigma^+}(j)]\}\, g(i-j) \tag{A.4}$$

where $\chi_A(l) = 1$ if $l \in A$ and 0 otherwise.

From assumption (a_2), we have that the first term in (A.4) can be estimated by

$$\sum_{i,j} \chi_{\Sigma^-}(i)\, g(i-j)\, \chi_{\Sigma^+}(j) \geqslant (1-b) \log \frac{d_{k+1}}{d_k} \tag{A.5}$$

where $b = b(\alpha, d_1)$ is such that $\lim_{d_1\to\infty} b = 0$.

The two other terms in (A.4) can be estimated as follows. Let us decompose $\hat{\mathcal{N}}$ according to the scale of its components, i.e.,

$$\hat{\mathcal{N}} = \bigcup_{l=1}^{k-1} \hat{\mathcal{N}_l} \tag{A.6}$$

where $\hat{\mathcal{N}_l} = \{J_j^l : j \in \hat{I}_{l+1}\}$, with J_i^l being an (l, i', δ, λ)-admissible neutral jump density for some $i' \in I_{l+1}^l(i)$.

In view of (A.6),

$$\hat{\Sigma} = \bigcup_{l=1}^{k-1} \hat{\Sigma}_l$$

with $\hat{\Sigma}_l \subset \bigcup_{j \in \hat{\mathcal{N}_l}} D(J)$ and by taking d_1 large enough, we have

$$\begin{aligned}\sum_{i,j} \chi_{\hat{\Sigma}^-}(i)\, g(i-j)\, \chi_{\Sigma^+}(j) &= \sum_{i,j} \sum_{l=1}^{k-1} \chi_{\hat{\Sigma}_l^-}(i)\, g(i-j)\, \chi_{\Sigma^+}(j) \\ &\leqslant 2 \sum_j \sum_{l=1}^{k-1} \frac{3}{2} d_l \sum_{i \in \hat{I}_{l+1}} \chi_{\Sigma^-}(i) \frac{1}{(i-j)^2} \chi_{\Sigma^+}(j) \\ &\leqslant C \sum_{l=1}^{k-1} \frac{d_l}{d_{l+1}} \log \frac{d_{k+1}}{d_k} \\ &\leqslant C' \log \frac{d_{k+1}}{d_k} \end{aligned} \tag{A.7}$$

where $C' = C'(\alpha, d_1)$ is a positive constant such that $C' \to 0$ as $d_1 \to \infty$.

Lemma 3.3 follows from (A.4), (A.5), and (A.7).

APPENDIX B

Proof of Lemma 4.1. We let $\sigma \in \mathscr{P}_{\mathscr{X}_0}$, $\sigma' \in \mathscr{P}_{\mathcal{N}/\mathscr{X}_0}$ be fixed. As in Appendix A, we decompose $\mathcal{N}/\mathscr{X}_0$ according to the scale of its components, i.e.,

$$\mathcal{N}/\mathscr{X}_0 = \bigcup_{l \geqslant 1} (\mathcal{N}/\mathscr{X}_0)_l$$

where each $J \in (\mathcal{N}/\mathscr{X}_0)_l$ is an (l, i', δ, λ)-admissible neutral jump density for some $i' \in I_{l+1}^l(i)$ and $i \in A_l$.

We have

$$\Delta H(\mathcal{N};\sigma,\sigma')$$

$$=\frac{1}{2}\left(\sum_{i,j}-\sum_{J\in\mathcal{N}}\sum_{i,j\in D(J)}\right)[(n_i^{J_\sigma}-n_j^{J_\sigma})^2-2(n_i^{J_\sigma}-n_j^{J_\sigma})(n_i^{J_{\sigma'}}-n_j^{J_{\sigma'}})]\;g(i,j)$$

$$\geqslant\sum_{i\in D(J_\sigma)}\sum_{l\geqslant 1}\sum_{J\in(\mathcal{N}/\mathcal{X}_0)_l}\sum_{j\in D(J)}[(n_i^{J_\sigma}-n_j^{J_\sigma})^2+2(n_i^{J_\sigma}-n_j^{J_\sigma})\,n_j^J]\;g(i,j)\quad\text{(B.1)}$$

From (3.8), $n_j^{J_\sigma}=\tilde{n}^{J_\sigma}$ for all $j\in D(J)$ and because of the neutrality of J and (3.8) we have

$$\sum_{j\in D(J)}n_j^J=\sum_k J(k)\sum_{k\leqslant j\leqslant i_{D(J)}^+}g(i,j)$$

$$\leqslant d_l\sum_k J(k)\;g(i,j)$$

[recall $i_D^{\pm}=\sup(\inf)\{i\in D\}$], which leads (B.1) to be bounded by

$$\geqslant\sum_{i\in D(J_\sigma)}\sum_{l\geqslant 1}d_l\sum_{J\in(\mathcal{N}/\mathcal{X}_0)_l}M_i(J,J_\sigma)$$

where

$$M_i(J,J_\sigma)=|n_i^{J_\sigma}-\tilde{n}^{J_\sigma}|\sum_{k\in D(J)}\left[\frac{1}{d_l}-J(k)\right]g(i,k)\qquad\text{(B.2)}$$

Lemma 4.1 follows if $M_i(J,J_\sigma)\geqslant 0$ for any $J\in(\mathcal{N}/\mathcal{X}_0)_l$, $l\geqslant 2$, and $i\in D(J_\sigma)$.

It follows from the construction given after Lemma 3.3 that

$$|n_i^{J_\sigma}-\tilde{n}^{J_\sigma}|=0\qquad\text{if}\quad\operatorname{dist}(i,D(J))<\tfrac{1}{3}d_{l+1}-d_l\qquad\text{(B.3)}$$

From (B.3) and neutrality of J we have

$$\sum_k J(k)\;g(i,k)\leqslant C\,|J|\left[\frac{1}{(i-i_{D(J)}^-)^2}-\frac{1}{(i-i_{D(J)}^+)^2}\right]$$

$$\leqslant C'\,|J|\,\frac{d_l}{|i-i_{D(J)}^-|^3}$$

$$\leqslant C_1\,|J|\,\frac{d_l}{d_{l+1}}\frac{1}{(i-i_{D(J)}^-)^2}\qquad\text{(B.4)}$$

for a fixed constant C_1. On the other hand,

$$\sum_{k \in D(J)} g(i, k) \geqslant C_2 \frac{d_l}{(i - i_{\bar{D}(J)})^2} \tag{B.5}$$

for another fixed constant C_2.

Conditions (B.4) and (B.5) imply that $M_i(J, J_\sigma) \geqslant 0$ for any $J \in (\mathcal{N}/\mathcal{X}_0)_l$, since

$$\frac{d_l}{d_{l+1}} |J| \leqslant \frac{d_l}{d_{l+1}} (\log d_l)^p$$

can be made arbitrarily small by choosing d_1 large enough.

This concludes the proof of Lemma 4.1.

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